ENTANGLEMENTS AND COMPOUND STATES IN QUANTUM INFORMATION THEORY

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Abstract. Quantum entanglements, describing truly quantum couplings, are studied and classified from the point of view of quantum compound states. We show that classical-quantum correspondences such as quantum encodings can be treated as d-entanglements leading to a special class of the separable compound states. The mutual information of the d-compound and entangled states lead to two different types of entropies for a given quantum state: the von Neumann entropy, which is achieved as the supremum of the information over all d-entanglements, and the dimensional entropy, which is achieved at the standard entanglement, the true quantum entanglement, coinciding with a d-entanglement only in the commutative case. The q-capacity of a quantum noiseless channel, defined as the supremum over all entanglements, is given as the logarithm of the dimensionality of the input von Neumann algebra. It can double the classical capacity, achieved as the supremum over all semi-quantum couplings (d-entanglements, or encodings), which is bounded by the logarithm of the dimensionality of a maximal Abelian subalgebra.

1. Introduction

Recently, the specifically quantum correlations, called in quantum physics entanglements, are used to study quantum information processes, in particular, quantum computation, quantum teleportation, quantum cryptography [1, 2, 3]. There have been mathematical studies of the entanglements in [4, 5, 6], in which the entangled state is defined by a compound state which can not be written as a convex combination $\sum_n \mu(n)\varsigma_n \otimes \varrho_n$ with any states ϱ_n and ς_n. However it is obvious that there exist several important applications with correlated states written as separable forms above. Such correlated, or entangled states have been also discussed in several contexts in quantum probability such as quantum measurement and filtering [7, 8], quantum compound state[9, 10] and lifting [11]. In this paper, we study the mathematical structure of quantum entangled states to provide a finer classification of quantum sates, and we discuss the informational degree of entanglement and entangled quantum mutual entropy.

We show that the pure entangled states can be treated as generalized compound states, the nonseparable states of quantum compound systems which are not representable by convex combinations of the product states.

The mixed compound states, defined as convex combinations by orthogonal decompositions of their input marginal states ϱ_0, have been introduced in [9] for studying the information in a quantum channel with the general output C*-algebra $\mathcal{A}$. This o-entangled compound state is a particular case of so called separable state of

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a compound system, the convex combination of the arbitrary product states which we call c-entangled. We shall prove that the o-entangled compound states are most informative among c-entangled states in the sense that the maximum of mutual information over all c-entanglements to the quantum system $(\mathcal{A}, \varrho)$ is achieved on the extreme o-entangled states, defined by a Schatten decomposition of a given state ϱ on $\mathcal{A}$. This maximum coincides with von Neumann entropy $S(\varrho)$ of the state ϱ, and it can also be achieved as the maximum of the mutual information over all couplings with classical probe systems described by a maximal Abelian subalgebra $\mathcal{A}^{\circ} \subseteq \mathcal{A}$. Thus the couplings described by c-entanglements of (quantum) probe systems $\mathcal{B}$ to a given system $\mathcal{A}$ don't give an advantage in maximizing the mutual information in comparison with the quantum-classical couplings, corresponding to the Abelian $\mathcal{B} = \mathcal{A}^{\circ}$. The achieved maximal information $S(\varrho)$ coincides with the classical entropy on the Abelian subalgebra $\mathcal{A}^{\circ}$ of a Schatten decomposition for ϱ, and is bounded by $\ln \operatorname{rank}\mathcal{A} = \ln \dim \mathcal{A}^{\circ}$, where $\operatorname{rank}\mathcal{A}$ is the rank of the von Neumann algebra $\mathcal{A}$ defined as the dimensionality of a maximal Abelian subalgebra. Due to $\dim \mathcal{A} \leq (\operatorname{rank}\mathcal{A})^2$, it is achieved on the normal central $\rho = (\operatorname{rank}\mathcal{A})^{-1} I$ only in the case of finite dimensional $\mathcal{A}$.

More general than o-entangled states, the d-entangled states, are defined as c-entangled states by orthogonal decomposition of only one marginal state on the probe algebra $\mathcal{B}$. They can give bigger mutual entropy for a quantum noisy channel than the o-entangled state which gains the same information as d-entangled extreme states in the case of a deterministic channel.

We prove that the truly (strongest) entangled states are most informative in the sense that the maximum of mutual entropy over all entanglements to the quantum system $\mathcal{A}$ is achieved on the quasi-compound state, given by an extreme entanglement of the probe system $\mathcal{B} = \mathcal{A}$ with coinciding marginals, called standard for a given ϱ. The standard entangled state is o-entangled only in the case of Abelian $\mathcal{A}$ or pure marginal state ϱ. The gained information for such extreme q-compound state defines another type of entropy, the quasi-entropy $S_q(\varrho)$ which is bigger than the von Neumann entropy $S(\varrho)$ in the case of non-Abelian $\mathcal{A}$ (and mixed ϱ.) The maximum of mutual entropy over all quantum couplings, described by true quantum entanglements of probe systems $\mathcal{B}$ to the system $\mathcal{A}$ is bounded by $\ln \dim\mathcal{A}$, the logarithm of the dimensionality of the von Neumann algebra $\mathcal{A}$, which is achieved on a normal tracial ρ in the case of finite dimensional $\mathcal{A}$. Thus the q-entropy $S_q(\varrho)$, which can be called the dimensional entropy, is the true quantum entropy, in contrast to the von Neumann rank entropy $S(\varrho)$, which is semi-classical entropy as it can be achieved as a supremum over all couplings with the classical probe systems $\mathcal{B}$. These entropies coincide in the classical case of Abelian $\mathcal{A}$ when $\operatorname{rank}\mathcal{A} = \dim \mathcal{A}$. In the case of non-Abelian finite-dimensional $\mathcal{A}$ the q-capacity $C_q = \ln \dim\mathcal{A}$ is achieved as the supremum of mutual entropy over all q-encodings (correspondences), described by entanglements. It is strictly bigger then the semi-classical capacity $C = \ln \operatorname{rank}\mathcal{A}$ of the identity channel, which is achieved as the supremum over usual encodings, described by the classical-quantum correspondences $\mathcal{A}^{\circ} \to \mathcal{A}$.

In this paper we consider the case of a discrete decomposable C*-algebra $\mathcal{A}$ for which the results are achieved by relatively simple proofs. The purely quantum case of a simple algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$, for which some proofs are rather obvious was considered in a short paper [12]. The general case of decomposable C*-algebra $\mathcal{A}$ to include the continuous systems, and will be published elsewhere.

2. Compound States and Entanglements

Let $\mathcal{H}$ denote the (separable) Hilbert space of a quantum system, and $\mathcal{L}(\mathcal{H})$ be the algebra of all linear bounded operators on $\mathcal{H}$. In order to include the classical discrete systems as a particular quantum case, we shall fix a decomposable subalgebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ of bounded observables $A \in \mathcal{A}$ of the form $A = \left[A(i)\,\delta_i^k\right]$, where $A(i) \in \mathcal{L}(\mathcal{H}_i)$ are arbitrary bounded operators in Hilbert subspaces $\mathcal{H}_i$, corresponding to an orthogonal decomposition $\mathcal{H} = \oplus_i \mathcal{H}_i$. A bounded linear functional $\varrho : \mathcal{A} \to \mathbf{C}$ is called a state on $\mathcal{A}$ if it is positive (i.e., $\varrho(A) \geq 0$ for any positive operator A in $\mathcal{A}$) and normalized $\varrho(I) = 1$ for the identity operator I in $\mathcal{A}$. A normal state can be expressed as

$$\varrho(A) = \mathrm{tr}_{\mathcal{G}} \kappa^\dagger A \kappa = \mathrm{tr} A\rho, \qquad A \in \mathcal{A} \tag{1}$$

where $\mathcal{G}$ is another separable Hilbert space, κ is a linear Hilbert-Schmidt operator from $\mathcal{G}$ to $\mathcal{H}$ and $\kappa^\dagger$ is the adjoint operator of κ from $\mathcal{H}$ to $\mathcal{G}$. This κ is called the amplitude operator, and it is called just the amplitude if $\mathcal{G}$ is one dimensional space $\mathbb{C}$, corresponding to the pure state $\varrho(A) = \kappa^\dagger A \kappa$ for a $\kappa \in \mathcal{H}$ with $\kappa^\dagger \kappa = \|\kappa\|^2 = 1$, in which case $\kappa^\dagger$ is the adjoint functional from $\mathcal{H}$ to $\mathbb{C}$. The density operator $\rho = \kappa\kappa^\dagger$ is uniquely defined by the condition $\rho \in \mathcal{A}$ as a decomposable trace class operator $\mathrm{P}_{\mathcal{A}} = \oplus \mathrm{P}_{\mathcal{A}}(i)$ with $\mathrm{P}_{\mathcal{A}}(i) \in \mathcal{L}(\mathcal{H}_i)$,

$$\nu(i) = \mathrm{tr}_{\mathcal{H}_i} \mathrm{P}_{\mathcal{A}}(i) \geq 0, \quad \sum_i \nu(i) = 1.$$

Thus the predual space $\mathcal{A}_*$ can be identified with the direct sum $\oplus \mathcal{T}(\mathcal{H}_i)$ of the Banach spaces $\mathcal{T}(\mathcal{H}_i)$ of trace class operators in $\mathcal{H}_i$ (the density operators $\mathrm{P}_{\mathcal{A}} \in \mathcal{A}_*$, $\mathrm{P}_{\mathcal{B}} \in \mathcal{B}_*$ of the states ϱ, ς on different algebras $\mathcal{A}$, $\mathcal{B}$ will be usually denoted by different letters ρ, σ corresponding to their Greek variations ϱ, ς.)

In general, $\mathcal{G}$ is not one dimensional, the dimensionality $\dim \mathcal{G}$ must be not less than $\mathrm{rank}\rho$, the dimensionality of the range $\mathrm{ran}\rho \subseteq \mathcal{H}$ of the density operator ρ. We shall equip it with an isometric involution $J = J^\dagger$, $J^2 = I$, having the properties of complex conjugation on $\mathcal{G}$,

$$J \sum \lambda_j \zeta_j = \sum \bar{\lambda}_j J \zeta_j, \quad \forall \lambda_j \in \mathbf{C}, \zeta_j \in \mathcal{G}$$

with respect to which $J\sigma = \sigma J$ for the positive and so self-adjoint operator $\sigma = \kappa^\dagger \kappa = \sigma^\dagger$ on $\mathcal{G}$. The latter can also be expressed as the symmetricity property $\tilde{\varsigma} = \varsigma$ of the state $\varsigma(B) = \mathrm{tr} B\sigma$ given by the real and so symmetric density operator $\bar{\sigma} = \sigma = \tilde{\sigma}$ on $\mathcal{G}$ with respect to the complex conjugation $\bar{B} = JBJ$ and the tilda operation ($\mathcal{G}$-transponation) $\tilde{B} = JB^\dagger J$ on the algebra $\mathcal{L}(\mathcal{G})$, and thus on any tilda invariant decomposable subalgebra $\mathcal{B} \subseteq \mathcal{L}(\mathcal{G})$ containing $\kappa^\dagger \mathcal{A} \kappa \ni \sigma$.

For example, $\mathcal{G}$ can be realized as a subspace of $l^2(\mathbf{N})$ of complex sequences $\mathbf{N} \ni n \mapsto \zeta(n) \in \mathbb{C}$, with $\sum_n |\zeta(n)|^2 < +\infty$ in the diagonal representation $\sigma = [\mu(n)\,\delta_n^m]$. The involution J can be identified with the complex conjugation $C\zeta(n) = \bar{\zeta}(n)$, i.e.,

$$C : \zeta = \sum_n |n\rangle \zeta(n) \mapsto C\zeta = \sum_n |n\rangle \bar{\zeta}(n)$$

in the standard basis $\{|n\rangle\} \subset \mathcal{G}$ of $l^2(\mathbf{N})$. In this case $\kappa = \sum \kappa_n \langle n|$ is given by orthogonal eigen-amplitudes $\kappa_n \in \mathcal{H}$, $\kappa_m^\dagger \kappa_n = 0$, $m \neq n$, normalized to the eigenvalues $\lambda(n) = \kappa_n^\dagger \kappa_n = \mu(n)$ of the density operator ρ such that $\rho = \sum \kappa_n \kappa_n^\dagger$ is a Schatten decomposition, i.e. the spectral decomposition of ρ into one-dimensional

orthogonal projectors. In any other basis the operator J is defined then by $J = U^{\dagger}CU$, where U is the corresponding unitary transformation. One can also identify $\mathcal{G}$ with $\mathcal{H}$ by $U\kappa_n = \lambda(n)^{1/2}|n\rangle$ such that the operator ρ is real and symmetric, $J\rho J = \rho = J\rho^{\dagger}J$ in $\mathcal{G} = \mathcal{H}$ with respect to the involution J defined in $\mathcal{H}$ by $J\kappa_n = \kappa_n$. Here U is an isometric operator $\mathcal{H} \to l^2(\mathbb{N})$ diagonalizing the operator ρ: $U\rho U^{\dagger} = \sum |n\rangle\lambda(n)\langle n|$. The amplitude operator $\kappa = \rho^{1/2}$ corresponding to $\mathcal{B} = \mathcal{A}$, $\sigma = \rho$ is called standard.

Given the amplitude operator κ, one can define not only the states ϱ and ς on the algebras $\mathcal{A} = \mathcal{L}(\mathcal{H})$ and $\mathcal{B} = \mathcal{L}(\mathcal{G})$ but also a pure entanglement state ϖ on the algebra $\mathcal{B} \otimes \mathcal{A}$ of all bounded operators on the tensor product Hilbert space $\mathcal{G} \otimes \mathcal{H}$ by

$$\varpi(B \otimes A) = \mathrm{tr}_{\mathcal{G}}\tilde{B}\kappa^{\dagger}A\kappa = \mathrm{tr}_{\mathcal{H}}A\kappa\tilde{B}\kappa^{\dagger}.$$

Indeed, thus defined ϖ is uniquely extended by linearity to a normal state on the algebra $\mathcal{B} \otimes \mathcal{A}$ generated by all linear combinations $C = \sum \lambda_j B_j \otimes A_j$ due to $\varpi(I \otimes I) = \mathrm{tr}\kappa^{\dagger}\kappa = 1$ and

$$\begin{aligned}\varpi\left(C^{\dagger}C\right) &= \sum_{i,k}\bar{\lambda}_i\lambda_k\mathrm{tr}_{\mathcal{G}}\tilde{B}_k\tilde{B}_i^{\dagger}\kappa^{\dagger}A_i^{\dagger}A_k\kappa \\ &= \sum_{i,k}\bar{\lambda}_i\lambda_k\mathrm{tr}_{\mathcal{G}}\tilde{B}_i^{\dagger}\kappa^{\dagger}A_i^{\dagger}A_k\kappa\tilde{B}_k = \mathrm{tr}_{\mathcal{G}}\chi^{\dagger}\chi \geq 0,\end{aligned}$$

where $\chi = \sum_j A_j\kappa\tilde{B}_j$. This state is pure on $\mathcal{L}(\mathcal{G} \otimes \mathcal{H})$ as it is given by an amplitude $\vartheta \in \mathcal{G} \otimes \mathcal{H}$ defined as

$$(\zeta \otimes \eta)^{\dagger}\vartheta = \eta^{\dagger}\kappa J\zeta, \quad \forall\zeta \in \mathcal{G}, \eta \in \mathcal{H},$$

and it has the states ϱ and ς as the marginals of ϖ:

$$\varpi(I \otimes A) = \mathrm{tr}_{\mathcal{H}}A\rho, \quad \varpi(B \otimes I) = \mathrm{tr}_{\mathcal{G}}B\sigma. \tag{2}$$

As follows from the next theorem for the case $\mathcal{F} = \mathbb{C}$, any pure state

$$\varpi(B \otimes A) = \vartheta^{\dagger}(B \otimes A)\vartheta, \quad B \in \mathcal{B}, A \in \mathcal{A}$$

given on $\mathcal{L}(\mathcal{G} \otimes \mathcal{H})$ by an amplitude $\vartheta \in \mathcal{G} \otimes \mathcal{H}$ with $\vartheta^{\dagger}\vartheta = 1$, can be achieved by a unique entanglement of its marginal states ς and ϱ.

Theorem 2.1. *Let $\varpi : \mathcal{B} \otimes \mathcal{A} \to \mathbb{C}$ be a compound state*

$$\varpi(B \otimes A) = \mathrm{tr}_{\mathcal{F}}\upsilon^{\dagger}(B \otimes A)\upsilon,$$

defined by an amplitude operator $\upsilon : \mathcal{F} \to \mathcal{G} \otimes \mathcal{H}$ on a separable Hilbert space $\mathcal{F}$ into the tensor product Hilbert space $\mathcal{G} \otimes \mathcal{H}$ with

$$\upsilon\upsilon^{\dagger} \in \mathcal{B} \otimes \mathcal{A}, \quad \mathrm{tr}_{\mathcal{F}}\upsilon^{\dagger}\upsilon = 1.$$

Then this state can be achieved as an entanglement

$$\varpi(B \otimes A) = \mathrm{tr}_{\mathcal{G}}\tilde{B}\kappa^{\dagger}(I \otimes A)\kappa = \mathrm{tr}_{\mathcal{F}\otimes\mathcal{H}}(I \otimes A)\kappa\tilde{B}\kappa^{\dagger} \tag{3}$$

of the states (2) with $\sigma = \kappa^{\dagger}\kappa$ and $\rho = \mathrm{tr}_{\mathcal{F}}\kappa\kappa^{\dagger}$, where κ is an amplitude operator $\mathcal{G} \to \mathcal{F} \otimes \mathcal{H}$ with

$$\kappa^{\dagger}(I \otimes \mathcal{A})\kappa \subset \mathcal{B}, \ \mathrm{tr}_{\mathcal{F}}\kappa\mathcal{B}\kappa^{\dagger} \subset \mathcal{A}. \tag{4}$$

The entangling operator κ is uniquely defined by $\tilde{\kappa}U = \upsilon$ up to a unitary transformation U of the minimal domain $\mathcal{F} = \mathrm{dom}\upsilon$.

Proof. Without loss of generality we can assume that the space $\mathcal{F}$ is equipped with an isometric involution J as well as the space $\mathcal{G}$ is equipped with J . The entangling operator κ can be defined then as $\kappa = (U \otimes I)\,\tilde{\upsilon}$ by

$$(U\xi \otimes \eta)^{\dagger}\kappa\zeta = (\xi \otimes \eta)^{\dagger}\tilde{\upsilon}\zeta := (J\zeta \otimes \eta)^{\dagger}\,\upsilon J\xi, \quad \forall \xi \in \mathcal{F}, \zeta \in \mathcal{G}, \eta \in \mathcal{H},$$

where U is arbitrary linear isometry in $\mathcal{F}$. Indeed, let $\{\xi_k\}$ be an orthonormal basis of $\mathcal{F}$**,** in which, say (but not necessary,) the density operator $\upsilon^{\dagger}\upsilon$ is diagonal, and $J : \mathcal{F} \to \mathcal{F}$ be the complex conjugation in this basis, $J\xi_k = \xi_k$, defining an isometric involution in $\mathcal{F}$. In general J is different from the complex conjugation C, given by $C|n\rangle = |n\rangle$ in the standard basis $\{|n\rangle; n \in \mathbb{N}\}$ if $\mathcal{F}$ is identified with a subspace $l^2(\mathbf{N})$ for the diagonal representation of $\upsilon^{\dagger}\upsilon$. Note that although the isometric transformation $U = \sum_k |k\rangle\xi_k^{\dagger}$ of the arbitrary basis $\{\xi_k\} \subset \mathcal{F}$ into $\{|k\rangle\} \subset l^2(\mathbf{N})$ is also arbitrary, it can be always considered as real with respect to C and $J = U^{\dagger}CU$, in the sense $\bar{U} := CUJ = U$, and so $\tilde{U} := CU^{\dagger}J = U^{\dagger}$. Defining $\kappa = \sum \kappa_n \langle n|$ in the standard basises of $\mathcal{F}$ and $\mathcal{G}$ as the block-matrix $\sum_{kn} |k\rangle \otimes \psi_k(n)\langle n|$ transposed to $\sum_{kn} |n\rangle \otimes \psi_k(n)\langle k|$, where the amplitudes $\psi_k(n) \in \mathcal{H}$ are given by the matrix elements $\eta^{\dagger}\psi_k(n) = \left(\langle n| \otimes \eta^{\dagger}\right)\upsilon\xi_k$, we obtain

$$\begin{aligned}
\mathrm{tr}_{\mathcal{G}}\tilde{B}\kappa^{\dagger}(I \otimes A)\,\kappa &= \sum_{n,m} \langle n|\,\tilde{B}\,|m\rangle\,\psi_k^{\dagger}(m)\,A\psi_k(n) \\
&= \sum_{n,m} \psi_k^{\dagger}(m)\,\langle m|\,B\,|n\rangle\,A\psi_k(n) = \mathrm{tr}_{\mathcal{F}}\upsilon^{\dagger}(B \otimes A)\,\upsilon\ .
\end{aligned}$$

Hence $\kappa : \mathcal{G} \to \mathcal{F} \otimes \mathcal{H}$, defined by $\kappa_n = \sum |k\rangle\psi_k(n)$ as the transposed to $\upsilon U^{\dagger} = \upsilon\tilde{U} \equiv \tilde{\kappa}$, is the required entangling operator of the form $\kappa = (U \otimes I)\,\tilde{\upsilon}$ with $\kappa^{\dagger}\kappa = \sigma = \mathrm{tr}_{\mathcal{H}}\upsilon\upsilon^{\dagger}$ and $\mathrm{tr}_{\mathcal{F}}\kappa\kappa^{\dagger} = \rho = \mathrm{tr}_{\mathcal{G}}\upsilon\upsilon^{\dagger}$. Moreover, it satisfies the conditions (4) as $\omega = \upsilon\upsilon^{\dagger} \in \mathcal{B} \otimes \mathcal{A}$ and thus

$$\kappa^{\dagger}(I \otimes A)\,\kappa = \mathrm{tr}_{\mathcal{H}}(I \otimes A)\,\omega \in \mathcal{B},\ \mathrm{tr}_{\mathcal{F}}\kappa\tilde{B}\kappa^{\dagger} = \mathrm{tr}_{\mathcal{G}}(B \otimes I)\,\omega \in \mathcal{A}.$$

The uniqueness follows from the obvious isometricity of the families

$$\left\{\sum_k |k\rangle\eta^{\dagger}\psi_k(n) : n \in \mathbb{N}, \eta \in \mathcal{H}\right\}, \quad \left\{\sum_k \eta^{\dagger}\psi_k(n)\,\xi_k^{\dagger} : n \in \mathbb{N}, \eta \in \mathcal{H}\right\}$$

of vectors $\left(I \otimes \eta^{\dagger}\right)\kappa|n\rangle$ in $\mathcal{F} \subseteq l^2(\mathbf{N})$ and of $\left(\langle n| \otimes \eta^{\dagger}\right)\upsilon$ in $\mathcal{F}^{\dagger}$ which follows from

$$\mathrm{tr}_{\mathcal{G}}|n\rangle\langle m|\kappa^{\dagger}\left(I \otimes \eta\eta^{\dagger}\right)\kappa = \mathrm{tr}_{\mathcal{F}}\upsilon^{\dagger}\left(|m\rangle\langle n| \otimes \eta\eta^{\dagger}\right)\upsilon.$$

Thus they are unitary equivalent in the minimal space $\mathcal{F}$. So the entangling operator κ is defined in the minimal $\mathcal{F}$ up to the unitary equivalence, corresponding to the arbitrary of the unitary operator U in $\mathcal{F}$, intertwining the involutions C and J. □

Note that the entangled state (3) is written as

$$\varpi(B \otimes A) = \mathrm{tr}_{\mathcal{G}}\tilde{B}\pi(A) = \mathrm{tr}_{\mathcal{H}}A\pi_*\left(\tilde{B}\right),$$

where $\pi(A) = \kappa^{\dagger}(I \otimes A)\,\kappa$, bounded by $\|A\|\,\sigma \in \mathcal{B}_*$ for any $A \in \mathcal{L}(\mathcal{H})$, is in the predual space $\mathcal{B}_* \subset \mathcal{B}$ of all trace-class operators in $\mathcal{G}$, and $\pi_*(B) = \mathrm{tr}_{\mathcal{F}}\kappa B\kappa^{\dagger}$,

bounded by $\|B\| \rho \in \mathcal{A}_*$, is in $\mathcal{A}_* \subset \mathcal{A}$. The map π is the Steinspring form [18] of the general completely positive map $\mathcal{A} \to \mathcal{B}_*$, written in the eigen-basis $\{|k\rangle\} \subset \mathcal{F}$ of the density operator $\upsilon^\dagger \upsilon$ as

$$\pi(A) = \sum_{m,n} |m\rangle \kappa_m^\dagger (I \otimes A) \kappa_n \langle n|, \quad A \in \mathcal{A} \tag{5}$$

while the dual operation π_* is the Kraus form [19] of the general completely positive map A $\to \mathcal{A}_*$, given in this basis as

$$\pi_*(B) = \sum_{n,m} \langle n| B |m\rangle \operatorname{tr}_{\mathcal{F}} \kappa_n \kappa_m^\dagger = \operatorname{tr}_{\mathcal{G}} \tilde{B} \omega. \tag{6}$$

It corresponds to the general form

$$\omega = \sum_{m,n} |n\rangle\langle m| \otimes \operatorname{tr}_{\mathcal{F}} \kappa_n \kappa_m^\dagger \tag{7}$$

of the density operator $\omega = \upsilon\upsilon^\dagger$ for the entangled state $\varpi(B \otimes A) = \operatorname{tr}(B \otimes A)\omega$ in this basis, characterized by the weak orthogonality property

$$\operatorname{tr}_{\mathcal{F}} \psi(m)^\dagger \psi(n) = \mu(n) \delta_n^m \tag{8}$$

in terms of the amplitude operators $\psi(n) = (I \otimes \langle n|) \tilde{\kappa} = \tilde{\kappa}_n$.

Definition 2.1. *The dual map $\pi_* : \mathcal{B} \to \mathcal{A}_*$ to a completely positive map $\pi : \mathcal{A} \to \mathcal{B}_*$, normalized as $\operatorname{tr}_{\mathcal{G}} \pi(I) = 1$, is called the quantum entanglement of the state $\varsigma = \pi(I)$ on $\mathcal{B}$ to the state $\varrho = \pi_*(I)$ on $\mathcal{A}$. The entanglement by*

$$\pi_*^\circ(A) = \rho^{1/2} A \rho^{1/2} = \pi^\circ(A) \tag{9}$$

of the state $\varsigma = \varrho$ on the algebra $\mathcal{B} = \mathcal{A}$ is called standard for the system $(\mathcal{A}, \varrho)$.

The standard entanglement defines the standard compound state

$$\varpi_0(B \otimes A) = \operatorname{tr}_{\mathcal{H}} \tilde{B} \rho^{1/2} A \rho^{1/2} = \operatorname{tr}_{\mathcal{H}} A \rho^{1/2} \tilde{B} \rho^{1/2}$$

on the algebra $\mathcal{A} \otimes \mathcal{A}$, which is pure, given by the amplitude $\vartheta_0 = \tilde{\kappa}_0$, where $\kappa_0 = \rho^{1/2}$ in the case of the simple algebra $\mathcal{A} = \mathcal{L}(\mathcal{H})$. In the general case of decomposable $\mathcal{A} = \oplus \mathcal{L}(\mathcal{H}_i)$ with the density operator $\rho = \oplus \rho(i)$ having more than one components $\rho(i) = \rho_i \nu(i)$ with $\nu(i) = \operatorname{tr} \rho(i) \neq 0$ and positive $\rho_i \in \mathcal{L}(\mathcal{H}_i)$, the standard state ϖ_0 is a mixture

$$\varpi_0(B \otimes A) = \sum_i \vartheta_0^{i\dagger} (B(i) \otimes A(i)) \vartheta_0^i \nu(i), \quad A, B \in \mathcal{A} \tag{10}$$

of such pure compound states given by the amplitudes $\vartheta_0^i \in \mathcal{H}_i \otimes \mathcal{H}_i$ with $\tilde{\vartheta}_0^i \tilde{\vartheta}_0^{i\dagger} = \rho_i$. The standard amplitudes $\vartheta_0^i \in \mathcal{H}_i \otimes \mathcal{H}_i$ for an orthogonal decomposition $\upsilon_0 = \sum_i \vartheta_0^i \xi_i^\dagger \nu(i)^{1/2}$ of the standard amplitude operator $\upsilon_0 : \mathcal{F}_0 \to \mathcal{H} \otimes \mathcal{H}$ are defined as $\tilde{\kappa}_0(i) / \|\tilde{\kappa}_0(i)\|$ by the entangling components $\kappa_0(i) = \rho(i)^{1/2}$ with

$$(\zeta_i \otimes \eta_i)^\dagger \tilde{\kappa}_0(i) = \eta_i^\dagger \kappa_0(i) J \zeta_i, \quad \forall \eta_i, \zeta_i \in \mathcal{H}_i.$$

[Example] In quantum physics the entangled states are usually obtained by a unitary transformation U of an initial disentangled state, described by the density operator $\sigma_0 \otimes \rho_0 \otimes \tau_0$ on the tensor product Hilbert space $\mathcal{G} \otimes \mathcal{H} \otimes \mathcal{K}$, that is,

$$\varpi(B \otimes A) = \operatorname{tr} U_1^\dagger (B \otimes A \otimes I) U_1 (\sigma_0 \otimes \rho_0 \otimes \tau_0).$$

In the simple case, when $\mathcal{K} = \mathbb{C}$, $\tau_0 = 1$, the joint amplitude operator υ is defined on the tensor product $\mathcal{F} = \mathcal{G} \otimes \mathcal{H}_0$ with $\mathcal{H}_0 = \mathrm{ran}\rho_0$ as $\upsilon = U_1 (\sigma_0 \otimes \rho_0)^{1/2}$. The entangling operator κ, describing the entangled state ϖ, is constructed as it was done in the proof of Theorem 1 by transponation of the operator $\upsilon U^\dagger$, where U is arbitrary isometric operator $\mathcal{F} \to \mathcal{G} \otimes \mathcal{H}_0$. The dynamical procedure of such entanglement in terms of the completely positive map $\pi_* : \mathcal{A} \to \mathcal{B}_*$ is the subject of Belavkin quantum filtering theory [17]. The quantum filtering dilation theorem [17] proves that any entanglement π can be obtained the unitary entanglement as the result of quantum filtering by tracing out some degrees of freedom of a quantum environment, described by the density operator τ_0 on the Hilbert space $\mathcal{K}$, even in the continuous time case.

3. C- and D-Entanglements and Encodings

The compound states play the role of joint input-output probability measures in classical information channels, and can be pure in quantum case even if the marginal states are mixed. The pure compound states achieved by an entanglement of mixed input and output states exhibit new, non-classical type of correlations which are responsible for the EPR type paradoxes in the interpretation of quantum theory. The mixed compound states on $\mathcal{B} \otimes \mathcal{A}$ which are given as the convex combinations

$$\varpi = \sum_n \varsigma_n \otimes \varrho_n \mu(n), \quad \mu(n) \geq 0, \ \sum_n \mu(n) = 1$$

of tensor products of pure or mixed normalized states $\varrho_n \in \mathcal{A}_*$, $\varsigma_n \in \mathcal{B}_*$ as in classical case, do not exhibit such paradoxical behavior, and are usually considered as the proper candidates for the input-output states in the communication channels. Such separable compound states are achieved by c-entanglements, the convex combinations of the primitive entanglements $B \mapsto \mathrm{tr}_{\mathcal{G}} B\omega_n$, given by the density operators $\omega_n = \sigma_n \otimes \rho_n$ of the product states $\varpi_n = \varsigma_n \otimes \varrho_n$:

$$\pi_*(B) = \sum_n \rho_n \mathrm{tr}_{\mathcal{G}} B\sigma_n \mu(n), \tag{11}$$

A compound state of this sort was introduced by Ohya [9] in order to define the quantum mutual entropy expressing the amount of information transmitted from an input quantum system to an output quantum system through a quantum channel, using a Schatten decomposition $\sigma = \sum_n \sigma_n \mu(n)$, $\sigma_n = |n\rangle\langle n|$ of the input density operator σ. It corresponds to a particular, diagonal type

$$\pi(A) = \sum_n |n\rangle \kappa_n^\dagger (I \otimes A) \kappa_n \langle n| \tag{12}$$

of the entangling map (5) in an eigen-basis $\{|n\rangle\} \in \mathcal{G}$ of the density operator σ, and is discussed in this section.

Let us consider a finite or infinite input system indexed by the natural numbers $n \in \mathbf{N}$. The associated space $\mathcal{G} \subseteq l^2(\mathbf{N})$ is the Hilbert space of the input system described by a quantum projection-valued measure $n \mapsto |n\rangle\langle n|$ on $\mathbb{N}$, given an orthogonal partition of unity $I = \sum |n\rangle\langle n| \in \mathcal{B}$ of the finite or infinite dimensional input Hilbert space $\mathcal{G}$. Each input pure state, identified with the one-dimensional density operator $|n\rangle\langle n| \in \mathcal{B}$ corresponding to the elementary symbol $n \in \mathbb{N}$, defines the elementary output state ϱ_n on $\mathcal{A}$. If the elementary states ϱ_n are pure, they are described by output amplitudes $\eta_n \in \mathcal{H}$ satisfying $\eta_n^\dagger \eta_n = 1 = \mathrm{tr}\rho_n$, where $\rho_n = \eta_n \eta_n^\dagger$ are the corresponding output one-dimensional density operators. If

these amplitudes are non-orthogonal $\eta_m^\dagger \eta_n \neq \delta_n^m$, they cannot be identified with the input amplitudes $|n\rangle$.

The elementary joint input-output states are given by the density operators $|n\rangle\langle n| \otimes \rho_n$ in $\mathcal{G} \otimes \mathcal{H}$. Their mixtures

$$\omega = \sum_n \mu(n) |n\rangle\langle n| \otimes \rho_n, \tag{13}$$

define the compound states on $\mathcal{B} \otimes \mathcal{A}$, given by the quantum correspondences $n \mapsto |n\rangle\langle n|$ with the probabilities $\mu(n)$. Here we note that the quantum correspondence is described by a classical-quantum channel, and the general d-compound state for a quantum-quantum channel in quantum communication can be obtained in this way due to the orthogonality of the decomposition (13), corresponding to the orthogonality of the Schatten decomposition $\sigma = \sum_n |n\rangle \mu(n) \langle n|$ for $\sigma = \mathrm{tr}_{\mathcal{H}} \omega$.

The comparison of the general compound state (7) with (13) suggests that the quantum correspondences are described as the diagonal entanglements

$$\pi_* (B) = \sum_n \mu(n) \langle n|B|n\rangle \rho_n, \tag{14}$$

They are dual to the orthogonal decompositions (12):

$$\pi(A) = \sum_n \mu(n) |n\rangle \eta_n^\dagger A \eta_n \langle n| = \sum_n |n\rangle \eta(n)^\dagger A \eta(n) \langle n|,$$

where $\eta(n) = \mu(n)^{1/2} \eta_n$. These are the entanglements with the stronger orthogonality

$$\psi(m) \psi(n)^\dagger = \rho(n) \delta_n^m, \tag{15}$$

for the amplitude operators $\psi(n) : \mathcal{F} \to \mathcal{H}$ of the decomposition $\upsilon = \sum_n |n\rangle \otimes \psi(n)$ in comparison with the orthogonality (8). The orthogonality (15) can be achieved in the following manner: Take in (5) $\kappa_n = |n\rangle \otimes \eta(n)$ with $\langle m|n\rangle = \delta_n^m$ so that

$$\kappa_m^\dagger (I \otimes A) \kappa_n = \mu(n) \eta_n^\dagger A \eta_n \delta_n^m$$

for any $A \in \mathcal{A}$. Then the strong orthogonality condition (15) is fulfilled by the amplitude operators $\psi(n) = \eta(n) \langle n| = \tilde{\kappa}_n$, and

$$\kappa^\dagger \kappa = \sum_n \mu(n) |n\rangle\langle n| = \sigma, \ \kappa\kappa^\dagger = \sum_n \eta(n) \eta(n)^\dagger = \rho.$$

It corresponds to the amplitude operator for the compound state (13) of the form

$$\upsilon = \sum_n |n\rangle \otimes \psi(n) U, \tag{16}$$

where U is arbitrary unitary operator from $\mathcal{F}$ onto $\mathcal{G}$, i.e. υ is unitary equivalent to the diagonal amplitude operator

$$\kappa = \sum_n |n\rangle\langle n| \otimes \eta(n)$$

on $\mathcal{F} = \mathcal{G}$ into $\mathcal{G} \otimes \mathcal{H}$. Thus, we have proved the following theorem in the case of pure output states $\rho_n = \eta_n \eta_n^\dagger$.

Theorem 3.1. *Let π be the operator (13), defining a d-compound state of the form*

$$\varpi(B \otimes A) = \sum_n \langle n|B|n\rangle \mathrm{tr}_{\mathcal{F}_n} \psi_n^\dagger A \psi_n \mu(n) \tag{17}$$

Then it corresponds to the entanglement by the orthogonal decomposition (12) mapping the algebra $\mathcal{A}$ into a diagonal subalgebra of $\mathcal{B}$.

Proof. Let $\oplus_n \mathcal{F}_n$ be the Hilbert orthogonal sum of the domains $\mathcal{F}_n$ for the amplitude operators ψ_n in (17) with an isometric involution $\oplus C_n$. In the case $\mathcal{F}_n = \mathbb{C}$ of the amplitudes $\psi_n \in \mathcal{H}$ corresponding to pure states ρ_n the involution $\oplus_n C_n$ is the componentwise complex conjugation in $\oplus_n \mathbb{C} \subseteq l^2(\mathbb{N})$; in the general case it is given by some isometric involutions C_n in the Hilbert spaces $\mathcal{F}_n$, which are equivalent to the ranges $\mathcal{H}_n = \rho_n \mathcal{H}$ of the density operators $\rho_n = \psi_n \psi_n^\dagger$ with the standard involutions in their eigen-representations, or contain these ranges. We can define the global output amplitude operator $\psi(n)$ on $\mathcal{F} = \oplus_n \mathcal{F}_n$ by

$$\psi(n) = \mu(n)^{1/2} \psi_n \epsilon_n^\dagger,$$

where $\epsilon_n : \mathcal{F}_n \to \mathcal{F}$ are the canonical orthogonal isometries, $\epsilon_m^\dagger \epsilon_n = I_n \delta_n^m$, and by (16) an amplitude operator $\upsilon : \mathcal{F} \to \mathcal{G} \otimes \mathcal{H}$ of the compound state (17), defining its density operator $\omega = \upsilon\upsilon^\dagger$ independently of the unitary transformation U of the Hilbert space onto $\oplus_n \mathcal{F}_n$.

The entangling operator $\kappa = \sum_n \kappa_n \langle n|$ is then defined by its components $\kappa_n \in \mathcal{F} \otimes \mathcal{H}$ of the form

$$\kappa_n = (\epsilon_n \otimes I) \tilde{\psi}_n \mu(n)^{1/2} = \tilde{\psi}(n),$$

Here $\tilde{\psi}_n$ are the amplitudes in $\mathcal{F}_n \otimes \mathcal{H}$ obtained from the operators $\psi_n : \mathcal{F}_n \to \mathcal{H}$ by

$$(\xi_n \otimes \eta)^\dagger \tilde{\psi}_n = \eta^\dagger \psi_n C_n \xi_n, \quad \forall \eta \in \mathcal{H}, \xi_n \in \mathcal{F}_n$$

In particular κ is the diagonal amplitude operator with the components $\kappa_n = \oplus_m \delta_n^m \tilde{\psi}(n)$ in $\oplus_m \mathcal{F}_m \otimes \mathcal{H}$:

$$\kappa = \sum_n \kappa_n \langle n| = \oplus_m \tilde{\psi}(m) \langle m|. \tag{18}$$

Thus the entanglement (6) corresponding to (17) is given by the dual to (12) diagonal map (14) with the density operators $\rho(n) = \psi(n)\psi(n)^\dagger = \mathrm{tr}_{\mathcal{F}} \kappa_n \kappa_n^\dagger$ normalized to the probabilities $\mu(n) = \kappa_n^\dagger \kappa_n$. □

Note that (2.9) defines the general form of a positive map on $\mathcal{A}$ with values in the simultaneously diagonal trace-class operators in A.

Definition 3.1. *A convex combination (11) of the primitive CP maps $\rho_n \varsigma_n$ is called c-entanglement, and is called d-entanglement, or quantum encoding if it has the diagonal form (14) on $\mathcal{B}$. The d-entanglement is called o-entanglement and compound state is called o-compound if all density operators ρ_n are orthogonal: $\rho_m \rho_n = \rho_n \rho_m$ for all m and n.*

Note that due to the commutativity of the operators $B \otimes I$ with $I \otimes A$ on $\mathcal{G} \otimes \mathcal{H}$, one can treat the correspondences as the nondemolition measurements [8] in $\mathcal{B}$ with respect to $\mathcal{A}$. So, the compound state is the state prepared for such measurements on the input $\mathcal{G}$. It coincides with the mixture of the states, corresponding to those after the measurement without reading the sent message. The set of all d-entanglements corresponding to a given Schatten decomposition of the input state σ on $\mathcal{B}$ is obviously convex with the extreme points given by the pure output states ρ_n

on $\mathcal{A}$, corresponding to a not necessarily orthogonal decompositions $\rho = \sum_n \rho(n)$ into one-dimensional density operators $\rho(n) = \mu(n)\rho_n$.

The Schatten decompositions $\rho = \sum_n \lambda(n)\rho_n$ correspond to the extreme d-entanglements, $\rho_n = \eta_n\eta_n^\dagger$, $\mu(n) = \lambda(n)$, characterized by orthogonality $\rho_m\rho_n = 0$, $m \neq n$. They form a convex set of d-entanglements with mixed commuting ρ_n for each Schatten decomposition of ρ. The orthogonal d-entanglements were used in [16] to construct a particular type of Accardi's transitional expectations [15] and to define the entropy in a quantum dynamical system via such transitional expectations.

The established structure of the general q-compound states suggests also the general form

$$\Phi_*(B, \varrho_0) = \mathrm{tr}_{\mathcal{F}_1} X^\dagger (B \otimes \rho_0) X = \mathrm{tr}_{\mathcal{G}} \left(\tilde{B} \otimes I\right) Y (I \otimes \rho_0) Y^\dagger$$

of transitional expectations $\Phi_* : \mathcal{B} \times \mathcal{A}_*^\circ \to \mathcal{A}_*$, describing the entanglements $\pi_* = \Phi_*(\varrho_0)$ of the states $\varsigma = \pi(I)$ to $\varrho = \pi_*(I)$ for each initial state $\varrho_0 \in \mathcal{A}_*^\circ$ with the density operator $\rho_0 \in \mathcal{A}^\circ \subseteq \mathcal{L}(\mathcal{H}_0)$ by $\pi_*(B) = \mathrm{tr}_{\mathcal{F}}\kappa(B \otimes I)\kappa^\dagger$, where $\kappa = X^\dagger (I \otimes \rho_0)^{1/2}$. It is given by an entangling transition operator $X : \mathcal{F}\otimes\mathcal{H} \to \mathcal{G}\otimes\mathcal{H}_0$, which is defined by a transitional amplitude operator $Y : \mathcal{H}_0 \otimes \mathcal{F} \to \mathcal{G} \otimes \mathcal{H}$ up to a unitary operator U in $\mathcal{F}$ as

$$(\zeta \otimes \eta_0)^\dagger X (U\xi \otimes \eta) = (\eta_0 \otimes J\xi)^\dagger Y^\dagger (J\zeta \otimes \eta).$$

The dual map $\Phi : \mathcal{A} \to \mathcal{B}_* \otimes \mathcal{A}^\circ$ is obviously normal and completely positive,

$$\Phi(A) = X(I \otimes A) X^\dagger \in \mathcal{B}_* \otimes \mathcal{A}^\circ,\ \forall A \in \mathcal{A}, \tag{19}$$

with $\mathrm{tr}_{\mathcal{G}}\Phi(I) = I^\circ$, and is called filtering map with the output states

$$\varsigma = \mathrm{tr}_{\mathcal{H}_0}\Phi(I)(I \otimes \rho_0)$$

in the theory of CP flows [17] over $\mathcal{A} = \mathcal{A}^\circ$. The operators Y normalized as $\mathrm{tr}_{\mathcal{F}} Y^\dagger Y = I^\circ$ describe $\mathcal{A}$-valued q-compound states

$$\mathrm{E}(B \otimes A) = \mathrm{tr}_{\mathcal{F}} Y^\dagger (B \otimes A) Y = \mathrm{tr}_{\mathcal{G}}\left(\tilde{B} \otimes I\right)\Phi(A),$$

defined as the normal completely positive maps $\mathcal{B} \otimes \mathcal{A} \to \mathcal{A}^\circ$ with $\mathrm{E}(I \otimes I) = I^\circ$.

If the $\mathcal{A}$-valued compound state has the diagonal form given by the orthogonal decomposition

$$\Phi(A) = \sum_n |n\rangle \mathrm{tr}_{\mathcal{F}}\Psi(n)^\dagger A\Psi(n)\langle n|, \tag{20}$$

corresponding to $Y = \sum_n |n\rangle \otimes \Psi(n)$, where $\Psi(n) : \mathcal{H}_0 \otimes \mathcal{F} \to \mathcal{H}$, it is achieved by the d-transitional expectations

$$\Phi_*(B, \varrho_0) = \sum_n \langle n|B|n\rangle \Psi(n)(\rho_0 \otimes I)\Psi(n)^\dagger.$$

The d-transitional expectations correspond to the instruments [20] of the dynamical theory of quantum measurements. The elementary filters

$$\Theta_n(A) = \frac{1}{\mu(n)}\mathrm{tr}_{\mathcal{F}}\Psi^\dagger(n) A\Psi(n), \quad \mu(n) = \mathrm{tr}\Psi(n)(\rho_0 \otimes I)\Psi^\dagger(n)$$

define posterior states $\varrho_n = \varrho_0 \Theta_n$ on $\mathcal{A}$ for quantum nondemolition measurements in $\mathcal{B}$, which are called indirect if the corresponding density operators ρ_n are nonorthogonal. They describe the posterior states with orthogonal

$$\rho_n = \Psi_n \left(\rho_0 \otimes I\right) \Psi_n^\dagger, \quad \Psi_n = \Psi\left(n\right)/\mu\left(n\right)^{1/2}$$

for all ρ_0 iff $\Psi\left(n\right)^\dagger \Psi\left(n\right) = \delta_n^m M\left(n\right)$.

4. Quantum Entropy via Entanglements

As it was shown in the previous section, the diagonal entanglements describe the classical-quantum encodings $\varkappa : \mathcal{B} \to \mathcal{A}_*$, i.e. correspondences of classical symbols to quantum, in general not orthogonal and pure, states. As we have seen in contrast to the classical case, not every entanglement can be achieved in this way. The general entangled states ϖ are described by the density operators $\omega = vv^\dagger$ of the form (7) which are not necessarily block-diagonal in the eigen-representation of the density operator σ, and they cannot be achieved even by a more general c-entanglement (11). Such nonseparable entangled states are called in [13] the quasicompound (q-compound) states, so we can call also the quantum nonseparable correspondences the quasi-encodings (q-encodings) in contrast to the d-correspondences, described by the diagonal entanglements.

As we shall prove in this section, the most informative for a quantum system $(\mathcal{A}, \varrho)$ is the standard entanglement $\pi_*^\circ = \pi_0$ of the probe system $(\mathcal{B}^\circ, \varsigma_0) = (\mathcal{A}, \varrho)$, described in (9). The other extreme cases of the self-dual input entanglements

$$\pi_*\left(A\right) = \sum_n \rho\left(n\right)^{1/2} A \rho\left(n\right)^{1/2} = \pi\left(A\right),$$

are the pure c-entanglements, given by the decompositions $\rho = \sum \rho\left(n\right)$ into pure states $\rho\left(n\right) = \eta_n \eta_n^\dagger \mu\left(n\right)$. We shall see that these c-entanglements, corresponding to the separable states

$$\omega = \sum_n \eta_n \eta_n^\dagger \otimes \eta_n \eta_n^\dagger \mu\left(n\right), \tag{21}$$

are in general less informative then the pure d-entanglements, given in an orthonormal basis $\{\eta_n^\circ\} \subset \mathcal{H}$ by

$$\pi^\circ\left(A\right) = \sum_n \eta_n^\circ \eta_n^\dagger A \eta_n \eta_n^{\circ\dagger} \mu\left(n\right) \neq \pi_*^\circ\left(A\right).$$

Now, let us consider the entangled mutual information and quantum entropies of states by means of the above three types of compound states. To define the quantum mutual entropy, we need the relative entropy [21, 22] of the compound state ϖ with respect to a reference state φ on the algebra $\mathcal{A} \otimes \mathcal{B}$. In our discrete case of the decomposable algebras it is defined by the density operators $\omega, \phi \in \mathcal{B} \otimes \mathcal{A}$ of these states as

$$S\left(\varpi, \varphi\right) = \mathrm{tr}\omega\left(\ln \omega - \ln \phi\right). \tag{22}$$

It has a positive value $S\left(\varpi, \varphi\right) \in [0, \infty]$ if the states are equally normalized, say (as usually) $\mathrm{tr}\omega = 1 = \mathrm{tr}\phi$, and it can be finite only if the state ϖ is absolutely continuous with respect to the reference state φ, i.e. iff $\varpi\left(E\right) = 0$ for the maximal null-orthoprojector $E\phi = 0$.

The mutual information $I_{\mathcal{A},\mathcal{B}}(\varpi)$ of a compound state ϖ achieved by an entanglement $\pi_* : \mathcal{B} \to \mathcal{A}_*$ with the marginals

$$\varsigma(B) = \varpi(B \otimes I) = \mathrm{tr}_{\mathcal{G}} B\sigma, \; \varrho(A) = \varpi(I \otimes A) = \mathrm{tr}_{\mathcal{H}} A\rho$$

is defined as the relative entropy (22) with respect to the product state $\varphi = \varsigma \otimes \varrho$:

$$I_{\mathcal{A},\mathcal{B}}(\varpi) = \mathrm{tr}\omega\left(\ln\omega - \ln(\sigma \otimes I) - \ln(I \otimes \rho)\right). \tag{23}$$

Here the operator ω is uniquely defined by the entanglement π_* as its density in (6), or the $\mathcal{G}$-transposed to the operator $\tilde{\omega}$ in

$$\pi(A) = \kappa^\dagger (I \otimes A)\kappa = \mathrm{tr}_{\mathcal{H}} A\tilde{\omega}.$$

This quantity describes an information gain in a quantum system $(\mathcal{A}, \varrho)$ via an entanglement π_* of another system $(\mathcal{B}, \varsigma)$. It is naturally treated as a measure of the strength of an entanglement, having zero value only for completely disentangled states, corresponding to $\varpi = \varsigma \otimes \varrho$.

Proposition 4.1. *Let $\pi_*^\circ : \mathcal{B}^\circ \to \mathcal{A}_*$ be an entanglement π_*° of a state $\varsigma_0 = \pi^\circ(I)$ on a discrete decomposable algebra $\mathcal{B}^\circ \subseteq \mathcal{L}(\mathcal{G}_0)$ to the state $\varrho = \pi_*^\circ(I)$ on $\mathcal{A}$, and $\pi_* = \pi_*^\circ \mathrm{K}$ be an entanglement defined as the composition with a normal completely positive unital map $\mathrm{K} : \mathcal{B} \to \mathcal{B}^\circ$. Then $I_{\mathcal{A},\mathcal{B}}(\varpi) \le I_{\mathcal{A},\mathcal{B}^\circ}(\varpi_0)$, where ϖ, ϖ_0 are the compound states achieved by π_*° , π_* respectively. In particular, for any c-entanglement π_* to $(\mathcal{A}, \varsigma)$ there exists a not less informative d-entanglement $\pi_*^\circ = \varkappa$ with an Abelian $\mathcal{B}^\circ$, and the standard entanglement $\pi_0(A) = \rho^{1/2} A \rho^{1/2}$ of $\varsigma_0 = \varrho$ on $\mathcal{B}^\circ = \mathcal{A}$ is the maximal one in this sense.*

Proof. The first follows from the monotonicity property [21]

$$\varpi = \mathrm{K}_*\varpi_0, \varphi = \mathrm{K}_*\varphi_0 \Rightarrow S(\varpi, \varphi) \le S(\varpi_0, \varphi_0) \tag{24}$$

of the general relative entropy on a von Neuman algebra $\mathcal{M}$ with respect to the predual K_* to any normal completely positive unital map $\mathrm{K} : \mathcal{M} \to \mathcal{M}^\circ$. It should be applied to the ampliation $\mathrm{K}(B \otimes A) = \mathrm{K}(B) \otimes A$ of the CP map K from $\mathcal{B} \to \mathcal{B}^\circ$ to $\mathcal{B} \otimes \mathcal{A} \to \mathcal{B}^\circ \otimes \mathcal{A}$, with the compound state $\mathrm{K}_*\varpi_0 = \varpi_0(\mathrm{K} \otimes \mathrm{I})$ (I denotes the identity map $\mathcal{A} \to \mathcal{A}$) corresponding to the entanglement $\pi_* = \pi_*^\circ \mathrm{K}$ and $\mathrm{K}_*\varphi_0 = \varsigma \otimes \varrho$, $\varsigma = \varsigma_0 \mathrm{K}$ corresponding to $\varphi_0 = \varsigma_0 \otimes \varrho$.

This monotonicity property proves in particular that for any separable compound state on $\mathcal{B} \otimes \mathcal{A}$, which is prepared by a c-entanglement (11), there exists a diagonal entanglement π_*° to the system $(\mathcal{A}, \varrho)$with the same, or even bigger information gain (23). One can take even a classical system $(\mathcal{B}^\circ, \varsigma_0)$, say the diagonal sublagebra $\mathcal{B}^\circ$ on $\mathcal{G}_0 = l^2(\mathbf{N})$ with the state ς_0, induced by the measure ν, and consider the classical-quantum correspondence (encoding)

$$\pi_*^\circ(B^\circ) = \sum_n \beta(n)\rho_n \nu(n), \quad B^\circ = \sum_n |n\rangle \beta(n) \langle n|,$$

prescribing the states $\varrho_n(A) = \mathrm{tr} A\rho_n$ to the letters n with the probabilities $\nu(n)$. The information gain

$$I_{\mathcal{A},\mathcal{B}^\circ}(\varpi_0) = \sum_n \mu(n)\,\mathrm{tr}\rho_n (\ln\rho_n - \ln\rho).$$

is equal or bigger then $I_{\mathcal{A},\mathcal{B}}(\varpi)$ corresponding to $\omega = \sum_n \sigma_n \otimes \rho_n \nu(n)$ because the entanglement (11) is represented as the composition $\pi_*^\circ \mathrm{K}$ with the CP map

$$\mathrm{K}(B) = \sum_n |n\rangle \varsigma_n(B) \langle n|, \quad B \in \mathcal{B}$$

into the diagonal algebra $\mathcal{B}^\circ$.

The inequality (24) can be also applied to the standard entanglement, corresponding to the compound state (10) on $\mathcal{A} \otimes \mathcal{A} = \oplus_{i,k} \mathcal{A}(i) \otimes \mathcal{A}(k)$, where $\mathcal{A}(i) = \mathcal{L}(\mathcal{H}_i)$. It is described by the density operator

$$\omega_0 = \oplus_{i,k} \mathrm{P}_{\mathcal{A}\otimes\mathcal{A}}(i,k) = \oplus_i \vartheta_0^i \vartheta_0^{i\dagger} \nu(i), \tag{25}$$

with $\mathrm{P}_{\mathcal{A}\otimes\mathcal{A}}(i,k) = 0$, $i \neq k$ concentrated on the diagonal $\oplus_i \mathcal{A}(i) \otimes \mathcal{A}(i)$ of $\mathcal{A} \otimes \mathcal{A}$. The amplitudes $\vartheta_0^i \in \mathcal{H}_i \otimes \mathcal{H}_i$ are defined in (10) by orthogonal components $\kappa_0(i) = \rho(i)^{1/2}$ of the central decomposition $\kappa_0 = \sum_n |n\rangle \otimes \kappa_0(n)$ for the standard entangling operator $\kappa_0 : \mathcal{H} \to l^2(\mathbf{N}) \otimes \mathcal{H}$. Indeed, any entanglement $\pi_*(B) = \mathrm{tr}_{\mathcal{F}} \kappa B \kappa^\dagger$ as a normal CP map $\mathcal{B} \to \mathcal{A}$ normalized to the density operator $\rho = \mathrm{tr}_{\mathcal{F}} \kappa \kappa^\dagger$ can be represented as the composition $\pi_*^\circ \mathrm{K}$ of the standard entanglement $\pi_*^\circ = \pi_0$ on $(\mathcal{B}^\circ, \varsigma_0) = (\mathcal{A}, \varrho)$ and a normal unital CP map $\mathrm{K} : \mathcal{B} \to \mathcal{A}$. The CP map K is defined by $\rho^{1/2} \mathrm{K}(B) \rho^{1/2} = \pi_*(B)$ as

$$\mathrm{K}(B) = \mathrm{tr}_{\mathcal{F}_-} X^\dagger B X, \quad B \in \mathcal{B}$$

where X is an operator $\mathcal{F}_- \otimes \mathcal{H} \to \mathcal{G}$, $\mathrm{tr}_{\mathcal{F}_-} X^\dagger X = I$ such that $\kappa = (I^- \otimes \kappa_0) X^\dagger$ is an entangling operator for π. Thus the standard entanglement π_*° corresponds to the maximal mutual information. □

Note that any extreme d-entanglement

$$\pi_*^\circ(B) = \sum_n \langle n|B|n\rangle \rho_n^\circ \mu(n), \; B \in \mathcal{B}^\circ,$$

with $\rho = \sum_n \rho_n^\circ \mu(n)$ decomposed into pure normalized states $\rho_n^\circ = \eta_n \eta_n^\dagger$, is maximal among all c-entanglements in the sense $I_{\mathcal{A},\mathcal{B}}(\varpi_0) \geq I_{\mathcal{A},\mathcal{B}}(\varpi)$. This is because $\mathrm{tr}\rho_n^\circ \ln \rho_n^\circ = 0$, and therefore the information gain

$$I_{\mathcal{A},\mathcal{B}}(\varpi) = \sum_n \mu(n) \mathrm{tr}\rho_n (\ln \rho_n - \ln \rho).$$

with a fixed $\pi_*(I) = \rho$ achieves its supremum $-\mathrm{tr}_{\mathcal{H}} \rho \ln \rho$ at any such extreme d-entanglement π_*°. Thus the supremum of the information gain (23) over all c-entanglements to the system $(\mathcal{A}, \varrho)$ is the von Neumann entropy

$$S_{\mathcal{A}}(\varrho) = -\mathrm{tr}_{\mathcal{H}} \rho \ln \rho. \tag{26}$$

It is achieved on any extreme π_*°, for example given by the maximal Abelian subalgebra $\mathcal{B}^\circ \subseteq \mathcal{A}$, with the measure $\mu = \lambda$, corresponding to a Schatten decomposition $\rho = \sum_n \eta_n^\circ \eta_n^{\circ\dagger} \lambda(n)$, $\eta_m^{\circ\dagger} \eta_n^\circ = \delta_n^m$. The maximal value $\ln \mathrm{rank}\mathcal{A}$ of the von Neumann entropy is defined by the dimensionality $\mathrm{rank}\mathcal{A} = \dim \mathcal{B}^\circ$ of the maximal Abelian subalgebra of the decomposable algebra $\mathcal{A}$, i.e. by $\dim \mathcal{H}$.

Definition 4.1. *The maximal mutual information*

$$H_{\mathcal{A}}(\varrho) = \sup_{\pi_*(I)=\rho} I_{\mathcal{A},\mathcal{B}}(\varpi) = I_{\mathcal{A},\mathcal{B}^\circ}(\varpi_0), \tag{27}$$

achieved on $\mathcal{B}^{\circ}=\mathcal{A}$ *by the standard q-entanglement* $\pi_{*}^{\circ}(A)=\rho^{1/2}A\rho^{1/2}$ *for a fixed state* $\varrho(A)=\mathrm{tr}_{\mathcal{H}}A\rho$*, is called q-entropy of the state* ϱ*. The differences*

$$H_{\mathcal{B}|\mathcal{A}}(\varpi)=H_{\mathcal{B}}(\varsigma)-I_{\mathcal{A},\mathcal{B}}(\varpi)$$

$$S_{\mathcal{B}|\mathcal{A}}(\varpi)=S_{\mathcal{B}}(\varsigma)-I_{\mathcal{A},\mathcal{B}}(\varpi)$$

are respectively called the q-conditional entropy on $\mathcal{B}$ *with respect to* $\mathcal{A}$ *and the degree of disentanglement for the compound state* ϖ*.*

Obviously, $H_{\mathcal{B}|\mathcal{A}}(\varpi)$ is positive in contrast to the disentanglement $S_{\mathcal{B}|\mathcal{A}}(\varpi)$, having the positive maximal value $S_{\mathcal{B}|\mathcal{A}}(\varpi)=S_{\mathcal{B}}(\varsigma)$ in the case $\varpi=\varsigma\otimes\varrho$ of complete disentanglement, but which can achieve also a negative value

$$\inf_{\pi_{*}(I)=\rho}S_{\mathcal{B}|\mathcal{A}}(\varpi)=S_{\mathcal{A}}(\varsigma)-H_{\mathcal{A}}(\varrho)=\sum_{i}\nu(i)\,\mathrm{tr}_{\mathcal{H}_i}\rho_i\ln\rho_i \tag{28}$$

for the entangled states as the following theorem states. Here $\rho_i\in\mathcal{L}(\mathcal{H}_i)$ are the density operators of normalized factor-states $\varrho_i=\nu(i)^{-1}\varrho|\mathcal{L}(\mathcal{H}_i)$with $\nu(i)=\varrho(I^i)$, where I^i are the orthoprojectors onto $\mathcal{H}_i$. Obviously $H_{\mathcal{A}}(\varrho)=S_{\mathcal{A}}(\varrho)$ if the algebra $\mathcal{A}$ is completely decomposable, i.e. Abelian, and the maximal value $\ln\mathrm{rank}\mathcal{A}$ of $S_{\mathcal{A}}(\varrho)$ can be written as $\ln\dim\mathcal{A}$ in this case. The disentanglement $S_{\mathcal{B}|\mathcal{A}}(\varpi)$ is always positive in this case, as well as in the case of Abelian $\mathcal{B}$ when $H_{\mathcal{B}|\mathcal{A}}(\varpi)=S_{\mathcal{B}|\mathcal{A}}(\varpi)$.

Theorem 4.2. *Let* $\mathcal{A}$ *be the discrete decomposable algebra on* $\mathcal{H}=\oplus_i\mathcal{H}_i$*, with a normal state given by the density operator* $\rho=\oplus\rho(i)$*, and* $\mathcal{C}\subseteq\mathcal{A}$ *be its center with the state* $\nu=\varrho|\mathcal{C}$ *induced by the probability distribution* $\nu(i)=\mathrm{tr}\rho(i)$*. Then the q-entropy is given by the formula*

$$H_{\mathcal{A}}(\varrho)=\sum_{i}\left(\nu(i)\ln\nu(i)-2\mathrm{tr}_{\mathcal{H}_i}\rho(i)\ln\rho(i)\right), \tag{29}$$

i.e. $H_{\mathcal{A}}(\varrho)=H_{\mathcal{A}|\mathcal{C}}(\varrho)+H_{\mathcal{C}}(\nu)$*, where* $H_{\mathcal{C}}(\nu)=-\sum_i\nu(i)\ln\nu(i)=S_{\mathcal{C}}(\nu)$*, and*

$$H_{\mathcal{A}|\mathcal{C}}(\varrho)=-2\sum_{i}\nu(i)\,\mathrm{tr}_{\mathcal{H}_i}\rho_i\ln\rho_i=2S_{\mathcal{A}|\mathcal{C}}(\varrho),$$

with $\rho_i=\rho(i)/\nu(i)$*. It is positive,* $H_{\mathcal{A}}(\varrho)\in[0,\infty]$*, and if* $\mathcal{A}$ *is finite dimensional, it is bounded, with the maximal value* $H_{\mathcal{A}}(\varrho^{\circ})=\ln\dim\mathcal{A}$ *which is achieved on the tracial* $\rho^{\circ}=\oplus\rho_i^{\circ}\nu^{\circ}(i)$*,*

$$\rho_i^{\circ}=(\dim\mathcal{H}_i)^{-1}I^i,\quad \nu^{\circ}(i)=\dim\mathcal{A}(i)/\dim\mathcal{A},$$

where $\dim\mathcal{A}(i)=(\dim\mathcal{H}_i)^2$, $\dim\mathcal{A}=\sum_i\dim\mathcal{A}(i)$.

Proof. The q-entropy $H_{\mathcal{A}}(\varrho)$ is the supremum (27) of the mutual information (23) which is achieved on the standard entanglement, corresponding to the density operator (25) of the standard compound state (10) with $\mathcal{B}=\mathcal{A}$, $\sigma=\rho$. Thus $H_{\mathcal{A}}(\rho)=I_{\mathcal{A},\mathcal{A}}(\varpi_0)$, where

$$\begin{aligned}I_{\mathcal{A},\mathcal{A}}(\varpi_0)&=\mathrm{tr}_{\mathcal{H}\otimes\mathcal{H}}\omega_0\left(\ln\omega_0-\ln(\rho\otimes I)-\ln(I\otimes\rho)\right)\\&=\sum_{i}\nu(i)\ln\nu(i)-2\mathrm{tr}\rho\ln\rho=-\sum_{i}\nu(i)\left(\ln\nu(i)+2\mathrm{tr}_{\mathcal{H}_i}\rho_i\ln\rho_i\right).\end{aligned}$$

Here we used that $\mathrm{tr}\omega_0 \ln \omega_0 = \sum_i \nu(i) \ln \nu(i)$ due to

$$\omega_0 \ln \omega_0 = \oplus_{i,k} \mathrm{P}_{\mathcal{A}\otimes\mathcal{A}}(i,k) \ln \mathrm{P}_{\mathcal{A}\otimes\mathcal{A}}(i,k) = \oplus_i \nu(i) \vartheta_0^i \vartheta_0^{i\dagger} \ln \nu(i)$$

for the orthogonal diagonal decomposition (25) of ω_0 into one-dimensional orthoprojectors $\vartheta_0^i \vartheta_0^{i\dagger} = \mathrm{P}_{\mathcal{A}\otimes\mathcal{A}}(i,i)/\nu(i)$, and that $\mathrm{tr}\rho \ln \rho = \sum_i \nu(i)(\ln \nu(i) - S_{\mathcal{A}_i}(\varrho_i))$ due to

$$\rho \ln \rho = \oplus_i \mathrm{P}_{\mathcal{A}}(i) \ln \mathrm{P}_{\mathcal{A}}(i) = \oplus_i \nu(i) \rho_i (\ln \nu(i) + \ln \rho_i)$$

for the orthogonal decomposition $\rho = \oplus_i \nu(i) \mathrm{P}_{\mathcal{A}(i)}$, where $\mathrm{P}_{\mathcal{A}(i)} = \mathrm{P}_{\mathcal{A}}(i)/\nu(i) = \rho_i$, $\nu(i) = \mathrm{tr}\mathrm{P}_{\mathcal{A}}(i)$, $\mathrm{P}_{\mathcal{A}}(i) = \sum_k \mathrm{tr}_{\mathcal{H}} \mathrm{P}_{\mathcal{A}\otimes\mathcal{A}}(i,k) = \rho(i)$.

Thus $H_{\mathcal{A}}(\varrho) = H_{\mathcal{A}|\mathcal{C}}(\varrho) + H_{\mathcal{C}}(\nu) = 2S_{\mathcal{A}|\mathcal{C}}(\varrho) + S_{\mathcal{C}}(\nu)$ is positive, and it is bounded by

$$\begin{aligned} C_{\mathcal{A}} &= \sup_{\nu} \sum_i \nu(i) \left(2 \sup_{\varrho_i} S_{\mathcal{A}(i)}(\varrho_i) - \ln \nu(i) \right) \\ &= -\inf_{\nu} \sum_i \nu(i) (\ln \nu(i) - 2 \ln \dim \mathcal{H}_i) = \ln \dim \mathcal{A}. \end{aligned}$$

Here we used the fact that the supremum of von Neumann entropies

$$S_{\mathcal{A}(i)}(\varrho_i) = -\sum_i \mathrm{tr}_{\mathcal{H}_i} \rho_i \ln \rho_i$$

for the simple algebras $\mathcal{A}(i) = \mathcal{L}(\mathcal{H}_i)$ with $\dim \mathcal{A}(i) = (\dim \mathcal{H}_i)^2 < \infty$ is achieved on the tracial density operators $\rho_i = (\dim \mathcal{H}_i)^{-1} I^i \equiv \rho_i^\circ$, and the infimum of the relative entropy

$$S(\nu, \nu^\circ) = \sum_i \nu(i) (\ln \nu(i) - \ln \nu^\circ(i)),$$

where $\nu^\circ(i) = \dim \mathcal{A}(i) / \dim \mathcal{A}$, is zero, achieved at $\nu = \nu^\circ$. □

5. Quantum Channel and its Q-Capacity

Let $\mathcal{H}_0$ be a Hilbert space describing a quantum input system and $\mathcal{H}$ describe its output Hilbert space. A quantum channel is an affine operation sending each input state defined on $\mathcal{H}_0$ to an output state defined on $\mathcal{H}$ such that the mixtures of states are preserved. A deterministic quantum channel is given by a linear isometry $Y \colon \mathcal{H}_0 \to \mathcal{H}$ with $Y^\dagger Y = I^\circ$ (I° is the identify operator in $\mathcal{H}_0$) such that each input state vector $\eta \in \mathcal{H}_0$, $\|\eta\| = 1$ is transmitted into an output state vector $Y\eta \in \mathcal{H}$, $\|Y\eta\| = 1$. The orthogonal mixtures $\rho_0 = \sum_n \mu(n) \rho_n^\circ$ of the pure input states $\rho_n^\circ = \eta_n^\circ \eta_n^{\circ\dagger}$ are sent into the orthogonal mixtures $\rho = \sum_n \mu(n) \rho_n$ of the corresponding pure states $\rho_n = Y \rho_n^\circ Y^\dagger$.

A noisy quantum channel sends pure input states ϱ_0 into mixed ones $\varrho = \Lambda^*(\varrho_0)$ given by the dual Λ^* to a normal completely positive unital map $\Lambda : \mathcal{A} \to \mathcal{A}_0$,

$$\Lambda(A) = \mathrm{tr}_{\mathcal{F}_1} Y^\dagger A Y, \quad A \in \mathcal{A}$$

where Y is a linear operator from $\mathcal{H}_0 \otimes \mathcal{F}_+$ to $\mathcal{H}$ with $\mathrm{tr}_{\mathcal{F}_+} Y^\dagger Y = I^\circ$, and $\mathcal{F}_+$ is a separable Hilbert space of quantum noise in the channel. Each input mixed state ϱ_0 on $\mathcal{A}^\circ \subseteq \mathcal{L}(\mathcal{H}_0)$ is transmitted into an output state $\varrho = \varrho_0 \Lambda$ given by the density operator

$$\Lambda_*(\rho_0) = Y \left(\rho_0 \otimes I^+ \right) Y^\dagger \in \mathcal{A}_*$$

for each density operator $\rho_0 \in \mathcal{A}_*^\circ$, where I^+ is the identity operator in $\mathcal{F}_+$. Without loss of generality we can assume that the input algebra $\mathcal{A}^\circ$ is the smallest decomposable algebra, generated by the range $\Lambda(\mathcal{A})$ of the given map Λ.

The input entanglements $\varkappa : \mathcal{B} \to \mathcal{A}_*^\circ$ described as normal CP maps with $\varkappa(I) = \varrho_0$, define the quantum correspondences (q-encodings) of probe systems $(\mathcal{B}, \varsigma)$, $\varsigma = \varkappa^*(I)$, to $(\mathcal{A}^\circ, \varrho_0)$. As it was proven in the previous section, the most informative is the standard entanglement $\varkappa = \pi_*^\circ$, at least in the case of the trivial channel $\Lambda = \mathrm{I}$. This extreme input q-entanglement

$$\pi^\circ(A^\circ) = \rho_0^{1/2} A^\circ \rho_0^{1/2} = \pi_*^\circ(A^\circ), \quad A^\circ \in \mathcal{A}^\circ,$$

corresponding to the choice $(\mathcal{B}, \varsigma) = (\mathcal{A}^\circ, \varrho_0)$, defines the following density operator

$$\omega = (\mathrm{I} \otimes \Lambda)_* (\omega_q^\circ), \quad \omega_q^\circ = \bigoplus_i \left(\vartheta_0^\iota \vartheta_0^{\iota\dagger}\right) \nu_0(i) \tag{30}$$

of the input-output compound state $\varpi_q^\circ \Lambda$ on $\mathcal{A}^\circ \otimes \mathcal{A}$. It is given by the central decomposition $\rho_0 = \oplus \rho_{0i} \nu_0(i)$ of the density operator $\rho_0 \in \mathcal{A}_*^\circ = \oplus \mathcal{T}(\mathcal{H}_{0i})$, with the amplitudes $\vartheta_0^i \in \mathcal{H}_{0i}^{\otimes 2}$ defined by $\tilde{\vartheta}_0^\iota = \rho_{0i}^{1/2}$. The other extreme cases of the self-dual input entanglements, the pure c-entanglements corresponding to (21), can be less informative then the d-entanglements, given by the decompositions $\rho_0 = \sum \rho_0(n)$ into pure states $\rho_0(n) = \eta_n \eta_n^\dagger \mu(n)$. They define the density operators

$$\omega = (\mathrm{I} \otimes \Lambda)_* (\omega_d^\circ), \quad \omega_d^\circ = \sum_n \eta_n^\circ \eta_n^{\circ\dagger} \otimes \eta_n \eta_n^\dagger \mu_0(n), \tag{31}$$

of the $\mathcal{A}^\circ \otimes \mathcal{A}$-compound state $\varpi_d^\circ \Lambda$, which are known as the Ohya compound states $\varpi_o^\circ \Lambda$ [9] in the case

$$\rho_0(n) = \eta_n^\circ \eta_n^{\circ\dagger} \lambda_0(n), \quad \eta_m^{\circ\dagger} \eta_n^\circ = \delta_n^m,$$

of orthogonality of the density operators $\rho_0(n)$ normalized to the eigen-values $\lambda_0(n)$ of ρ_0. They are described by the input-output density operators

$$\omega = (\mathrm{I} \otimes \Lambda)_* (\omega_o^\circ), \quad \omega_o^\circ = \sum_n \eta_n^\circ \eta_n^{\circ\dagger} \otimes \eta_n^\circ \eta_n^{\circ\dagger} \lambda_0(n), \tag{32}$$

coinciding with (30) in the case of Abelian $\mathcal{A}^\circ$. These input-output compound states ϖ are achieved by compositions $\lambda = \pi^\circ \Lambda$, describing the entanglements λ^* of the extreme probe system $(\mathcal{B}^\circ, \varsigma_0) = (\mathcal{A}^\circ, \varrho_0)$ to the output $(\mathcal{A}, \varrho)$ of the channel.

If $\mathrm{K} : \mathcal{B} \to \mathcal{B}^\circ$ is a normal completely positive unital map

$$\mathrm{K}(B) = \mathrm{tr}_{\mathcal{F}_-} X^\dagger B X, \quad B \in \mathcal{B},$$

where X is a bounded operator $\mathcal{F}_- \otimes \mathcal{G}_0 \to \mathcal{G}$ with $\mathrm{tr}_{\mathcal{F}_-} X^\dagger X = I^\circ$, the compositions $\varkappa = \pi_*^\circ \mathrm{K}$, $\pi_* = \Lambda_* \varkappa$ are the entanglements of the probe system $(\mathcal{B}, \varsigma)$ to the channel input $(\mathcal{A}^\circ, \varrho_0)$ and to the output $(\mathcal{A}, \varrho)$ via this channel. The state $\varsigma = \varsigma_0 \mathrm{K}$ is given by

$$\mathrm{K}_*(\sigma_0) = X\left(I^- \otimes \sigma_0\right) X^\dagger \in \mathcal{B}_*$$

for each density operator $\sigma_0 \in \mathcal{B}_*^\circ$, where I^- is the identity operator in $\mathcal{F}_-$. The resulting entanglement $\pi_* = \lambda_* \mathrm{K}$ defines the compound state $\varpi = \varpi_0(\mathrm{K} \otimes \Lambda)$ on $\mathcal{B} \otimes \mathcal{A}$ with

$$\varpi_0(B^\circ \otimes A^\circ) = \mathrm{tr} \tilde{B}^\circ \pi^\circ(A^\circ) = \mathrm{tr} \upsilon_0^\dagger (B^\circ \otimes A^\circ) \upsilon_0.$$

on $\mathcal{B}^\circ \otimes \mathcal{A}^\circ$. Here $\upsilon_0 : \mathcal{F}_0 \to \mathcal{G}_0 \otimes \mathcal{H}_0$ is the amplitude operator, uniquely defined by the input compound state $\varpi_0 \in \mathcal{B}_*^\circ \otimes \mathcal{A}_*^\circ$ up to a unitary operator U° on $\mathcal{F}_0$, and

the effect of the input entanglement $\varkappa$ and the output channel Λ can be written in terms of the amplitude operator of the state ϖ as

$$\upsilon = (X \otimes Y) \left(I^{-} \otimes \upsilon_0 \otimes I^{+} \right) U$$

up to a unitary operator U in $\mathcal{F} = \mathcal{F}_{-} \otimes \mathcal{F}_0 \otimes \mathcal{F}_{+}$. Thus the density operator $\omega = \upsilon\upsilon^{\dagger}$ of the input-output compound state ϖ is given by $\varpi_0 (\mathrm{K} \otimes \Lambda)$ with the density

$$(\mathrm{K} \otimes \Lambda)_{*} (\omega_0) = (X \otimes Y) \, \omega_0 \, (X \otimes Y)^{\dagger} , \tag{33}$$

where $\omega_0 = \upsilon_0 \upsilon_0^{\dagger}$.

Let $\mathcal{K}_q$ be the convex set of normal completely positive maps $\varkappa : \mathcal{B} \rightarrow \mathcal{A}_{*}^{\circ}$ normalized as $\mathrm{tr}\varkappa (I) = 1$, and $\mathcal{K}_q^{\circ}$ be the convex subset $\{\varkappa \in \mathcal{K}_q : \varkappa (I) = \varrho_0\}$. Each $\varkappa \in \mathcal{K}_q^{\circ}$ can be decomposed as $\pi_{*}^{\circ}\mathrm{K}$, where $\pi_{*}^{\circ} = \pi^{\circ}$ is the standard entanglement on $(\mathcal{A}^{\circ}, \varrho_0)$, and K is a normal unital CP map $\mathcal{B} \rightarrow \mathcal{A}^{\circ}$. Further let $\mathcal{K}_c$ be the convex set of the maps $\varkappa$, dual to the input maps of the form (11), described by the combinations

$$\varkappa (B) = \sum_{n} \varsigma (B) \, \rho_0 (n) . \tag{34}$$

of the primitive maps $\varkappa_n : B \mapsto \varsigma_n (B) \, \rho_0 (n)$, and $\mathcal{K}_d$ be the subset of the diagonal decompositions

$$\varkappa (B) = \sum_{n} \langle n|B|n\rangle \rho_0 (n) . \tag{35}$$

As in the first case $\mathcal{K}_c^{\circ}$ and $\mathcal{K}_d^{\circ}$ denote the convex subsets corresponding to a fixed $\varkappa (I) = \varrho_0$, and each $\varkappa \in \mathcal{K}_c^{\circ}$ can be represented as $\pi_{*}^{\circ}\mathrm{K}$, where π_{*}° is a d-entanglement, which can be always be made pure by a proper choice of the CP map $\mathrm{K} : \mathcal{B} \rightarrow \mathcal{A}^{\circ}$. Furthermore let $\mathcal{K}_o$ ($\mathcal{K}_o^{\circ}$) be the subset of all decompositions (34) with orthogonal $\rho_0 (n)$ (and fixed $\sum_n \rho_0 (n) = \rho_0$):

$$\rho_0 (m) \, \rho_0 (n) = 0, \; m \neq n.$$

Each $\varkappa \in \mathcal{K}_o^{\circ}$ can be also represented as $\pi_{*}^{\circ}\mathrm{K}$, where π_{*}° is a diagonal pure o-entanglement $\mathcal{B} \rightarrow \mathcal{A}^{\circ}$.

Now, let us maximize the entangled mutual information for a given quantum channel Λ and a fixed input state ϱ_0 by means of the above four types of compound states. The mutual information (23) was defined in the previous section by the density operators of the compound state ϖ on $\mathcal{B}\otimes\mathcal{A}$, and the product-state $\varphi = \varsigma\otimes\varrho$ of the marginals ς, ϱ for ϖ. In each case

$$\varpi = \varpi_0 (\mathrm{K} \otimes \Lambda) , \quad \varphi = \varphi_0 (\mathrm{K} \otimes \Lambda) ,$$

where K is a CP map $\mathcal{B} \rightarrow \mathcal{B}^{\circ}$, ϖ_0 is one of the corresponding extreme compound states ϖ_q°, $\varpi_c^{\circ} = \varpi_d^{\circ}$, ϖ_o° on $\mathcal{A}^{\circ} \otimes \mathcal{A}^{\circ}$, and $\varphi_0 = \varrho_0 \otimes \varrho_0$. The density operator $\omega = (\mathrm{K} \otimes \Lambda)_{*} (\omega_0)$ is written in (33), and $\phi = \sigma \otimes \rho$ can be written as

$$\phi = \varkappa_{*} (I) \otimes \lambda_{*} (I) ,$$

where $\lambda_{*} = \Lambda_{*}\pi_{*}^{\circ}$.

Proposition 5.1. *The entangled mutual informations achieve the following maximal values*

$$\sup_{\varkappa\in\mathcal{K}_q^{\circ}} I_{\mathcal{A},\mathcal{B}} (\varpi) = I_q (\varrho_0, \Lambda) := I_{\mathcal{A},\mathcal{A}^{\circ}} \left(\varpi_q^{\circ}\Lambda \right) , \tag{36}$$

$$I_c(\varrho_0,\Lambda) = \sup_{\varkappa\in\mathcal{K}_c^\circ} I_{\mathcal{A},\mathcal{B}}(\varpi) = \sup_{\varpi_d^\circ} I_{\mathcal{A},\mathcal{A}^\circ}(\varpi_d^\circ\Lambda) = I_d(\varrho_0,\Lambda),$$

$$\sup_{\varkappa\in\mathcal{K}_o^\circ} I_{\mathcal{A},\mathcal{B}}(\varpi) = I_o(\varrho_0,\Lambda) := \sup_{\varpi_o^\circ} I_{\mathcal{A},\mathcal{A}^\circ}(\varpi_o^\circ\Lambda), \tag{37}$$

where $\varpi_\bullet^\circ$ are the corresponding extremal input entangled states on $\mathcal{A}^\circ\otimes\mathcal{A}^\circ$ with marginals ϱ_0. They are ordered as

$$I_q(\varrho_0,\Lambda) \geq I_c(\varrho_0,\Lambda) = I_d(\varrho_0,\Lambda) \geq I_o(\varrho_0,\Lambda). \tag{38}$$

Proof. Due to the monotonicity

$$I_{\mathcal{A},\mathcal{B}}(\varpi_d^\circ(\mathrm{K}\otimes\Lambda)) \leq I_{\mathcal{A},\mathcal{A}^\circ}(\varpi_d^\circ(\mathrm{I}\otimes\Lambda))$$

the supremum over all c-entanglements $\varkappa\in\mathcal{K}_c^\circ$ coinsides with the supremum over $\mathcal{K}_d^\circ\subset\mathcal{K}_c^\circ$ which is achieved on the pure d-entanglements on $(\mathcal{A}^\circ,\varrho_0)$ corresponding to the extreme compound states ϖ_d°. By the same monotonicity arguments we can get the equalities (36) and (37). The entanglements $\varkappa\in\mathcal{K}_q^\circ$ can be written as

$$\varkappa(B) = \sum_{m,n}\langle m|B|n\rangle\chi(m)\chi(n)^\dagger$$

in a basis $\{|n\rangle\}\subset\mathcal{G}$ for the Schatten decompositions $\sigma=\sum_n|n\rangle\mu(n)\langle n|$ corresponding to weakly orthogonal amplitude operators $\chi(n) = (\langle n|X\otimes I)(I^-\otimes\upsilon_0)$:

$$\mathrm{tr}\chi(m)\chi(n)^\dagger = \mu(n)\delta_n^m.$$

The maps $\varkappa\in\mathcal{K}_d^\circ$ can be written as

$$\varkappa(B) = \sum_n\langle n|B|n\rangle\chi(n)\chi(n)^\dagger$$

corresponding to stronger orthogonal amplitude operators

$$\chi(m)\chi(n)^\dagger = \rho_0(n)\delta_n^m,$$

defining not necessarily orthogonal decompositions $\rho_0=\sum_n\rho_0(n)$. The extreme maps $\varkappa\in\mathcal{K}_o^\circ$ can be written as

$$\varkappa(B) = \sum_n\langle n|B|n\rangle\chi(n)\chi(n)^\dagger$$

with amplitude operators $\chi(n)$, satisfying the second orthogonality condition

$$\chi(n)^\dagger\chi(m) = \mu(n)\tau_n^\circ\delta_n^m,$$

where τ_n° are density operators in $\mathcal{F}_0$ with the traces $\mathrm{tr}\tau_n^\circ=1$. Thus, the inequalities in (38) follow from $\mathcal{K}_q\supseteq\mathcal{K}_c\supseteq\mathcal{K}_d\supseteq\mathcal{K}_o$. □

We shall denote the maximal informations $I_c(\varrho_0,\Lambda) = I_d(\varrho_0,\Lambda)$ simply as $I(\varrho_0,\Lambda)$.

Definition 5.1. *The supremums*

$$C_q(\Lambda) = \sup_{\varkappa\in\mathcal{K}_q} I_{\mathcal{A},\mathcal{B}}(\varpi) = \sup_{\varrho_0} I_q(\varrho_0,\Lambda),$$

$$\sup_{\varkappa\in\mathcal{K}_c} I_{\mathcal{A},\mathcal{B}}(\varpi) = C(\Lambda) := \sup_{\varrho_0} I(\varrho_0,\Lambda), \tag{39}$$

$$C_o(\Lambda) = \sup_{\varkappa \in \mathcal{K}_o} I_{\mathcal{A},\mathcal{B}}(\varpi) = \sup_{\varrho_0} I_o(\varrho_0, \Lambda),$$

are called the q-, c- or d-, and o-capacities respectively for the quantum channel defined by a normal unital CP map $\Lambda : \mathcal{A} \to \mathcal{A}^\circ$.

Obviously the capacities (39) satisfy the inequalities

$$C_o(\Lambda) \leq C(\Lambda) \leq C_q(\Lambda).$$

Theorem 5.2. *Let $\Lambda(A) = Y^\dagger AY$ be a unital CP map $\mathcal{A} \to \mathcal{A}^\circ$ describing a quantum deterministic channel. Then*

$$I(\varrho_0, \Lambda) = I_o(\varrho_0, \Lambda) = S(\varrho_0), \quad I_q(\varrho_0, \Lambda) = S_q(\varrho_0),$$

where $S_q(\varrho_0) = H_{\mathcal{A}^\circ}(\varrho_0)$, and thus in this case

$$C(\Lambda) = C_o(\Lambda) = \ln \mathrm{rank}\mathcal{A}^\circ, \quad C_q(\Lambda) = \ln \dim \mathcal{A}^\circ.$$

Proof. It was proved in the previous section for the case of the identity channel $\Lambda = \mathrm{I}$, and thus it is also valied for any isomorphism Λ described by a unitary operator Y. In the case of non-unitary Y we can use the identity

$$\mathrm{tr} Y\left(\rho_0 \otimes I^+\right) Y^\dagger \ln Y\left(\rho_0 \otimes I^+\right) Y^\dagger = \mathrm{tr} R\left(\omega_0 \otimes I^+\right) \ln R\left(\omega_0 \otimes I^+\right),$$

where $R = Y^\dagger Y$. Due to this $S(\varrho_0 \Lambda) = -\mathrm{tr} R\left(\rho_0 \otimes I^+\right) \ln R\left(\rho_0 \otimes I^+\right)$, and $S(\varpi_0(\mathrm{K} \otimes \Lambda)) =$

$$-\mathrm{tr}(S \otimes R)\left(I^- \otimes \omega_0 \otimes I^+\right) \ln (S \otimes R)\left(I^- \otimes \omega_0 \otimes I^+\right),$$

where $S = X^\dagger X$. Thus $S(\varrho_0 \Lambda) = S(\varrho_0)$, $S(\varpi_0(\mathrm{K} \otimes \Lambda)) = S(\varpi_0(\mathrm{K} \otimes \mathrm{I}))$ if $Y^\dagger Y = I$, and

$$\begin{aligned} I_{\mathcal{A},\mathcal{B}}(\varpi_0(\mathrm{K} \otimes \Lambda)) &= S(\varsigma_0 \mathrm{K}) + S(\varrho_0) - S(\varpi_0(\mathrm{K} \otimes \mathrm{I})) \\ &\leq S(\varsigma_0) + S(\varrho_0) - S(\varpi_0) = I_{\mathcal{A}^\circ,\mathcal{B}^\circ}(\varpi_0) \end{aligned}$$

for any normal unital CP map $\mathrm{K} : \mathcal{B} \to \mathcal{B}^\circ$ and a compound state ϖ_0 on $\mathcal{B}^\circ \otimes \mathcal{A}^\circ$. The supremum (36), which is achieved at the standard entanglement, corresponding to $\varpi_0 = \varpi_q$, coincides with q-entropy $H_{\mathcal{A}^\circ}(\varrho_0)$, and the supremum (37), coinciding with $S_{\mathcal{A}^\circ}(\varrho_0)$, is achieved for a pure o-entanglement, corresponding to $\varpi_0 = \varpi_o$ given by any Schatten decomposition for ρ_0. Moreover, the entropy $H_{\mathcal{A}^\circ}(\varrho_0)$ is also achieved by any pure d-entanglement, corresponding to $\varpi_0 = \varpi_d$ given by any extreme decomposition for ρ_0, and thus is the maximal mutual information $I(\varrho_0, \Lambda)$ in the case of deterministic Λ. Thus the capacity $C(\Lambda)$ of the deterministic channel is given by the maximum $C_o = \ln \dim \mathcal{H}_0$ of the von Neumann entropy $S_{\mathcal{A}^\circ}$, and the q-capacity $C_q(\Lambda)$ is equal $C_{\mathcal{A}^\circ} = \ln \dim \mathcal{A}^\circ$. $\square$

In the general case d-entanglements can be more informative than o-entanglements as it can be shown on an example of a quantum noisy channel for which

$$I(\varrho_0, \Lambda) > I_o(\varrho_0, \Lambda), \quad C(\Lambda) > C_o(\Lambda).$$

The last equalities of the above theorem will be related to the work on entropy by Voiculescu [25].

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